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BCJ numerators from differential operator of multidimensional residue

Gang Chen[1,2,3,a], **Tianheng Wang**[2,4,5]

[1] Department of Physics, Zhejiang Normal University, Jinhua, China
[2] Department of Physics and Astronomy, Uppsala University, Uppsala, Sweden
[3] Centre for Research in String Theory, School of Physics and Astronomy, Queen Mary University of London, London, UK
[4] Department of Physics, Nanjing University, Nanjing, China
[5] Institut für Physik, Humboldt-Universität zu Berlin, Berlin, Germany

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Abstract In previous works, we devised a differential operator for evaluating typical integrals appearing in the Cachazo–He–Yuan (CHY) forms and in this paper we further streamline this method. We observe that at tree level, the number of free parameters controlling the differential operator depends solely on the number of external lines, after solving the constraints arising from the scattering equations. This allows us to construct a reduction matrix that relates the parameters of a higher-order differential operator to those of a lower-order one. The reduction matrix is theory-independent and can be obtained by solving a set of explicitly given linear conditions. The repeated application of such reduction matrices eventually transforms a given tree-level CHY-like integral to a prepared form. We also provide analytic expressions for the parameters associated with any such prepared form at tree level. We finally give a compact expression for the multidimensional residue for any CHY-like integral in terms of the reduction matrices. We adopt a dual basis projector which leads to the CHY-like representation for the non-local Bern–Carrasco–Johansson (BCJ) numerators at tree level in Yang–Mills theory. These BCJ numerators are efficiently computed by the improved method involving the reduction matrix.

1 Introduction

Scattering amplitudes in a number of theories can be packaged in the compact expressions known as the Cachazo–He–Yuan (CHY) forms [1–3]. The CHY forms are originally proposed for tree-level scattering amplitudes and later generalized to loop levels [4–13]. In the CHY form, the scattering amplitude is represented as a contour integral around the solutions to the scattering equations [1–3], which can be transformed to a polynomial form [14,15]. Such contour integrals can be evaluated using the integration rules and the cross-ratio method at tree and loop levels [16–19]. Systematic methods for computing these integrals are based on the analysis of multidimensional residues on the isolated solutions of the scattering equations. One method for computing multidimensional residues involving the Groebner basis or the H-basis is discussed in [20,21]. A useful Mathematica package for computing such residues is given in [22].

In [23,24] Cheung, Xu and the current authors proposed a method for evaluating the CHY forms using a differential operator and studied the combinatoric properties of the scattering equations. This method bypasses the need for solving the scattering equations and leads to the analytic evaluation of a particular class of CHY forms, called the prepared forms. In this paper, we further streamline the method at tree level by relating a generic CHY-like expression to a prepared form. A crucial observation in our approach is that the number of independent parameters appearing in such a differential operator is always $(n-3)!$ where n is the number of external lines, regardless of the order of the operator. As will become clear in later discussions, this observation allows us to develop a method that relates higher-order differential operators with lower-order ones, through the reduction matrices for each factor of the terms in the Pfaffian expansion. Our method is theory independent and it maintains the factorized form of CHY integrand. Due to the two advantages, the CHY integral is evaluated efficiently.

As a particular application of our method, we study the construction of the Bern–Carrasco–Johansson (BCJ) numerators [25] from the CHY forms. The color-kinematic duality is found to hold in a number of theories [25–36] and extensive studies have been dedicated to computing the BCJ numerators [37–41]. In [38,42], the twistor string theory have been

[a] e-mail: gang.chern@gmail.com

studied to extract the local Bern–Carrasco–Johansson (BCJ) [25] numerators. The CHY forms can also be used to study the BCJ numerators and in [43] the local BCJ numerators are constructed. In this paper, we extract the non-local BCJ numerators in the minimal basis at tree level from the CHY forms, by introducing a dual basis projector. This way the BCJ numerators also take CHY-like expressions and can be easily studied using the differential operator and the reduction matrix.

2 Preliminary: review of differential operator method

Here we briefly summarize our method for computing the multidimensional residue. Let $g_1, g_2, \ldots, g_k$ be homogeneous polynomials in complex variables $z_1, z_2, \ldots, z_k$. If their common zeros lie on a single *isolated* point p (for homogeneous polynomials, the point p is the origin), for a holomorphic function $\mathcal{R}(z_i)$ in a neighborhood of p, we conjecture that a differential operator $\mathbb{D}$ computes the residue of $\mathcal{R}$ at p as follows

$$\mathrm{Res}_{\{(g_1),\ldots,(g_k)\},p}[\mathcal{R}] \equiv \oint \frac{dz_1 \wedge \cdots \wedge dz_k}{g_1 \cdots g_k}\mathcal{R} = \mathbb{D}^{(m)}[\mathcal{R}]\Big|_{z_i \to 0}, \tag{1}$$

where $\mathbb{D}^{(m)}$ is a differential operator of order-m and takes the following form,

$$\mathbb{D}^{(m)} = \sum_{\{r_i\}_m} a_{r_1,r_2,\ldots,r_k} \partial^{r_1,r_2,\ldots,r_k} . \tag{2}$$

Here $\partial^{r_1,r_2,\ldots,r_k} = (\frac{\partial}{\partial z_1})^{r_1}(\frac{\partial}{\partial z_2})^{r_2}\cdots(\frac{\partial}{\partial z_k})^{r_k}$ and r_i's are non-negative integers satisfying the Frobenius equation $\sum_{i=1}^k r_i = m \equiv \sum_{i=1}^k \deg(g_i) - k$. The coefficients $a_{r_1,r_2,\ldots,r_k}$ are constants independent of z_i's, determined uniquely by two sets of constraints arising from: 1. the local duality theorem [44] and 2. the intersection number of the divisors $D_i = (g_i)$. Detailed discussions on these constraints can be found in [23]. This conjecture applies to any such multidimensional residues around an isolated pole.

The CHY form for a tree-level scattering amplitude or a loop-level integrand is an integral on a Riemann sphere completely localized by the scattering equations. Equivalently, it is a multi-dimensional residue around the common zeros of the scattering equations. The tree-level scattering equations for n external particles read

$$\sum_{j\neq i} \frac{k_i \cdot k_j}{\sigma_i - \sigma_j} = 0\,, \quad i \in [2, n-2]\,, \tag{3}$$

where we have already taken care of the $SL(2,\mathbb{C})$ conformal symmetry by fixing $\sigma_1 \to 0, \sigma_{n-1} \to 1, \sigma_n \to \infty$. A simple transformation found in [14,15] takes (3) to the polynomial ones

$$h_t = \left(\sum_{2\leqslant i_1<i_2<\cdots<i_t\leqslant n-1} s_{i_1 i_2\ldots i_t n}\sigma_{i_1}\ldots\sigma_{i_t} \right)\Bigg|_{\sigma_{n-1}\to\sigma_0}, \quad t \in [1, n-3]\,, \tag{4}$$

where $s_{i_1\ldots i_t n} = \frac{1}{2}(k_{i_1}+\cdots+k_{i_t}+k_n)^2$. Here we have introduced an auxiliary variable σ_0, which fomally renders the polynomials homogeneous for the above differential operator to apply. At $\sigma_0 \to 1$, the two versions of scattering equations, (3) and (4), are equivalent.[1] The Jacobian of the above transformation is given by a Vandermonde determinant $J_n(\sigma) = \prod_{1\leq r<t\leq n-1}(\sigma_t - \sigma_r)$.

Adopting the polynomial scattering equations, a tree-level n-point amplitude is schematically given by a combination of the following CHY-like integrals

$$\mathcal{I}_n(P, h_0) = \oint_{h_1=\cdots=h_{n-3}=\sigma_0-1=0} \frac{d\sigma_2 \wedge \cdots \wedge d\sigma_{n-2} \wedge d\sigma_0}{h_1 \ldots h_{n-3}(\sigma_0 - 1)} \frac{P(\sigma)}{h_0(\sigma)}, \tag{5}$$

where the integrand is a rational function specific to the underlying theory. Its explicit expressions in different contexts can be found in [1–3]. For our purpose, we only note that $h_0(\sigma)$ is a homogeneous polynomial and factorizes into products of degree-one polynomials. In addition to the $(n-3)$ polynomial scattering equations, $\sigma_0 - 1 = 0$ is also imposed to localize the auxiliary variable.

The global residue theorem allows us to consider the residue around the solution of $h_1 = \cdots = h_{n-3} = h_0 = 0$ instead[2]

$$\mathcal{I}_n(P, h_0) = -\oint_{h_1=\cdots=h_{n-3}=h_0=0} \frac{d\sigma_2 \wedge \cdots \wedge d\sigma_{n-2} \wedge d\sigma_0}{h_1 \ldots h_{n-3}h_0(\sigma)} \frac{P(\sigma)}{(\sigma_0 - 1)}. \tag{6}$$

The aforementioned differential operator then applies to (6) as follows

$$\mathcal{I}_n(P, h_0) = -\left[\mathbb{D}^{(m)}_{h_0} \frac{P(\sigma)}{(\sigma_0 - 1)}\right]\Bigg|_{\sigma\to 0}, \tag{7}$$

[1] The polynomial scattering equations are equivalent to the original ones at tree level. At loop levels, there are extra solutions, which is beyond the scope of this paper.

[2] Poles at infinity can in principle exist. Detailed discussions on poles at infinity are given in [23].

where $\sigma \to 0$ is simply a shorthand for $\sigma_r \to 0, r \in \{2, \ldots, n-2, 0\}$. Namely all σ_i's are taken to zero after the action of the differential operator. The differential operator $\mathbb{D}_{h_0}^{(m)}$ takes the form given in (2) with the parameters $a_{r_2,r_3,\ldots,r_{n-2},r_0}$ and

$$\partial^{r_2,r_3,\ldots,r_{n-2},r_0} = \left(\frac{\partial}{\partial\sigma_2}\right)^{r_2}\left(\frac{\partial}{\partial\sigma_3}\right)^{r_3} \ldots \left(\frac{\partial}{\partial\sigma_{n-2}}\right)^{r_{n-2}}\left(\frac{\partial}{\partial\sigma_0}\right)^{r_0}.$$

The parameters $a_{r_2,r_3,\ldots,r_{n-2},r_0}$ are determined by the polynomial scattering equations h_j $(j \in [1, n-3])$ and h_0. As the scattering equations are universal for all CHY-like integrals, we only specify h_0 in the labels of the differential operator. The order of this operator is again $m = 0 + 1 + \cdots + (n-4) + d_{h_0} - 1$, with d_{h_0} denoting the degree of h_0. We note that the scattering equations $h_j = 0$ $(j \in [1, n-3])$ and $\sigma_0 - 1 = 0$ have multiple common solutions. Namely (5) has multiple poles. The differential operator in (7) gives the sum of (5) evaluated at each solution, by the global residue theorem. For details, see [23]. In particular, when $h_0 = \sigma_i$, the CHY-like integral is studied in [24] and the corresponding differential operator is worked out analytically.

3 Reduction matrix and evaluation of CHY integrals

In this section, we propose a method for relating two differential operators associated with two CHY-like integrals. To be more precise, the two CHY-like integrals of the form (5) share the numerator $P(\sigma)$ and their respective h_0 and h_0' are related as $h_0(\sigma) = h_0'(\sigma)q(\sigma)$ with $q(\sigma)$ being also a polynomial. In this case, the a-coefficients in the two corresponding differential operators can be related by a matrix, which we call the *reduction matrix*. This leads to a systematic evaluation of any tree-level CHY-like integral, which is given in a factorized form.

3.1 Canonical coefficients in differential operator

Consider the differential forms below

$$\Gamma = \frac{P(\sigma)d\sigma_2 \wedge \cdots \wedge d\sigma_{n-2} \wedge d\sigma_0}{h_1 \ldots h_{n-3}h_0}, \tag{8}$$

whose residue at the origin is the same as the CHY-like integral of the form (5). Recall that the corresponding differential operator $\mathbb{D}_{h_0}^{(m)}$ takes the form of (2) as follows

$$\mathbb{D}_{h_0}^{(m)} = \sum_{\{r\}_m} a_{r_2,\ldots,r_{n-2},r_0}\partial^{r_2,\ldots,r_{n-2},r_0}. \tag{9}$$

Since the polynomial scattering equations $\{h_1, \ldots, h_{n-3}\}$ are universal, we can always solve the local duality conditions [23] arising from these polynomials first. These conditions read

$$\mathbb{D}_{h_0}^{(m)}\left[q_j(\sigma)h_j(\sigma)\right] = 0, \qquad j = 1, 2, \ldots n-3, \tag{10}$$

where $q_j(\sigma)$ scans over all the monomials in σ's of the degree $\deg(q_j) = m - j$. Substituting (9) into (10), we have

$$\sum_{i_1<i_2<\cdots<i_t}\left[\prod_{l=2}^{n-1}(r_l + v_l)!\right] s_{i_1i_2\ldots i_ti_n}a_{r_2+v_2,r_3+v_3,\ldots,r_{n-2}+v_{n-2},r_0+v_0} = 0, \tag{11}$$

where we always identify $r_{n-1} = r_0$ and $v_{n-1} = v_0$. The summation is taken over the subsets $\{i_1, \ldots, i_t\} \subset \{2, \ldots, n-1\}$ with $t \in [1, n-3]$. Here r_s's are non-negative integers and v_l's are defined as

$$v_l = \begin{cases} 1, & \text{if } l \in \{i_1, i_2, \ldots, i_t\} \\ 0, & \text{if } l \notin \{i_1, i_2, \ldots, i_t\} \end{cases}. \tag{12}$$

For a given n, the number of the a-coefficients and the number of local duality conditions both grow as d_{h_0} increases. However, we observe that the number of independent a-coefficients after solving the Eq. (11) is always $(n-3)!$, regardless of m.[3] This allows us to choose $(n-3)!$ a-coefficients as a basis and expand the rest on this basis.

For the purpose of this paper, we find a particularly convenient basis choice as follows,

$$\{a_{\gamma(0),\ldots,\gamma(n-4),(d_{h_0}-1)}|\gamma \in S_{n-3}\}, \tag{13}$$

where S_{n-3} denotes the permutations. Throughout this paper, the a-coefficients in the above set is called the *canonical coefficients*. The differential operator can then be rewritten only in the canonical coefficients

$$\mathbb{D}_{h_0}^{(m)} = \sum_{i=1}^{(n-3)!} a_{\gamma_i(0),\ldots,\gamma_i(n-4),d_{h_0}-1}\mathcal{D}_i^{(m)}, \tag{14}$$

where each γ_i denotes a different permutation. The non-canonical a-coefficients are expanded into the canonical ones

$$a_{r_2,\ldots,r_{n-3},r_0} = \sum_{i=1}^{(n-3)!} c_i^{r_2,\ldots,r_{n-3},r_0}a_{\gamma_i(0),\ldots,\gamma_i(n-4),(d_{h_0}-1)}, \tag{15}$$

[3] This is an observation from a large number of examples, both analytic and numeric. We don't have a proof for this observation at the moment.

with the coefficients $c_i^{r_2,\ldots,r_{n-3},r_0}$ obtained by solving (11). Collecting the canonical coefficients, we have

$$\mathcal{D}_i^{(m)} = \partial^{\gamma_i(0),\ldots,\gamma_i(n-4),(d_{h_0}-1)} + \sum_{\substack{\sum r_j=m,\\ \{r_j\}\notin S_{n-3}}} c_i^{r_2,\ldots,r_{n-3},r_0}\, \partial^{r_2,\ldots,r_{n-2},r_0}\,. \tag{16}$$

In (14), the differential operator $\mathbb{D}_{h_0}^{(m)}$ is expanded into the basis spanned by $\mathcal{D}_i^{(m)}$, $i\in[1,(n-3)!]$. We note that $\mathcal{D}_i^{(m)}$'s are determined solely by the order m and the scattering equations, independent of the actual form of h_0.

3.2 Reduction matrix

We now study the relation between the higher- and lower-order operators. Consider the meromorphic forms Γ and Γ' below

$$\Gamma = \frac{P(\sigma)d\sigma_2\wedge\cdots\wedge d\sigma_{n-2}\wedge d\sigma_0}{h_1\ldots h_{n-3}h_0}\,,\qquad \Gamma' = \frac{P(\sigma)d\sigma_2\wedge\cdots\wedge d\sigma_{n-2}\wedge d\sigma_0}{h_1\ldots h_{n-3}h_0'}, \tag{17}$$

where $h_0(\sigma)=h_0'(\sigma)q(\sigma)$ with $q(\sigma)$ also being a polynomial of degree d_q. Let $\mathbb{D}_{h_0}^{(m)}$ and $\mathbb{D}_{h_0'}^{(m-d_q)}$ denote their corresponding differential operators respectively. For an arbitrary homogeneous polynomial $P(\sigma)$ of degree $\deg(P)\leqslant m-d_q$, we must have

$$\left[\mathbb{D}_{h_0}^{(m)}\frac{q(\sigma)P(\sigma)}{\sigma_0-1}\right]\Bigg|_{\sigma\to 0} = \left[\mathbb{D}_{h_0'}^{(m-d_q)}\frac{P(\sigma)}{\sigma_0-1}\right]\Bigg|_{\sigma\to 0}. \tag{18}$$

Plugging in the solutions of the respective non-canonical a-coefficients on both sides and expressing both differential operators in terms of their canonical coefficients only, the equation above becomes

$$\sum_{i=1}^{(n-3)!} a_{\gamma_i(0),\ldots,\gamma_i(n-4),d_{h_0}-1}\left[\mathcal{D}_i^{(m)}\frac{q(\sigma)P(\sigma)}{\sigma_0-1}\right]\Bigg|_{\sigma\to 0} = \sum_{i=1}^{(n-3)!} a_{\gamma_i(0),\ldots,\gamma_i(n-4),d_{h_0}-d_q-1}\left[\mathcal{D}_i^{(m-d_q)}\frac{P(\sigma)}{\sigma_0-1}\right]\Bigg|_{\sigma\to 0}. \tag{19}$$

For this equation to hold for an arbitrary $P(\sigma)$, the coefficients of the surviving derivatives $\partial^{r_2,\ldots,r_{n-3},r_0}P$ with $r_2+\cdots+r_{n-3}+r_0<\deg(P)$ must be the same on both sides. This leads to linear relations between the two sets of canonical a-coefficients, which can be written in the matrix form

$$\begin{pmatrix}\vdots\\ a_{\gamma(0),\ldots,\gamma(n-4),d_{h_0}-1}\\ \vdots\end{pmatrix} = M_{q(\sigma)}^{(n,m)}\begin{pmatrix}\vdots\\ a_{\gamma(0),\ldots,\gamma(n-4),d_{h_0}-d_q-1}\\ \vdots\end{pmatrix}. \tag{20}$$

We name the matrix $M_{q(\sigma)}^{(n,m)}$ the *reduction matrix*. The reduction matrix depends only on the factor $q(\sigma)$ and the orders of the differential operators while it knows nothing about the specific expression of the factor h_0'. We note that although the reduction matrix depends on $q(\sigma)$, its entries are only functions of momenta.

The reduction process can be performed repeatedly. Typically h_0 in a CHY form is completely factorized as $h_0=q^{(1)}q^{(2)}\ldots q^{(d_{h_0})}$, where each $q^{(r)}=\sigma_i-\sigma_j$ is a degree-one polynomial in σ's. As a result, the canonical coefficients in $\mathbb{D}_{h_0}^{(m)}$ can eventually be related to those in an operator of order $m_0\equiv m-(d_{h_0}-1)$. For such a degree-one $q=\sigma_i-\sigma_j$, it is easy to check that (19) yields a simple relation between reduction matrices $M_q^{(n,m)}=\frac{1}{m-m_0}M_q^{(n,m_0+1)}$. For notational brevity, we define $M_q^{(n,m_0+1)}\equiv M_q^{(n)}$. Hence we have

$$\begin{pmatrix}\vdots\\ a_{\gamma(0),\ldots,\gamma(n-4),d_{h_0}-1}\\ \vdots\end{pmatrix} = \frac{M_{q^{(1)}}^{(n)}M_{q^{(2)}}^{(n)}\ldots M_{q^{(d_{h_0}-1)}}^{(n)}}{(d_{h_0}-1)!}\begin{pmatrix}\vdots\\ a_{\gamma(0),\ldots,\gamma(n-4),0}\\ \vdots\end{pmatrix}. \tag{21}$$

We note that the ordering of these $q^{(r)}$ factors do not affect the eventual evaluation of the CHY-like integral. The choice of $(d_{h_0}-1)$ factors for the reduction process is also irrelevant, although certain choices might be more convenient in particular cases.

The inverse of the reduction matrix is linear in the variables in the subscript, namely[4]

[4] The relation (19) can be schematically rewritten as $L_q\cdot\mathbf{a}_{d_{h_0}-1}=R\cdot\mathbf{a}_{d_{h_0}-d_q-1}$ where $\mathbf{a}_{d_{h_0}-1}$ and $\mathbf{a}_{d_{h_0}-d_q-1}$ are the two column vectors on the left and right sides of (20) respectively. L_q and R are matrices following directly from (19) and we note the matrix R is independent of q. Hence $M_q=L_q^{-1}\cdot R$. For $q=\sigma_{r_1}-\sigma_{r_2}$, (19) yields $L_{\sigma_{r_1}-\sigma_{r_2}}=L_{\sigma_{r_1}}-L_{\sigma_{r_2}}$. Moreover, since $M_{\sigma_{r_1}-\sigma_{r_2}}=L_{\sigma_{r_1}-\sigma_{r_2}}^{-1}\cdot R$ and $M_{\sigma_{r_i}}=L_{\sigma_{r_i}}^{-1}\cdot R$, $i=1,2$, taking the inverse of the three reduction matrices, we obtain the relation (22).

$$M^{(n)}_{\sigma_{r_1}-\sigma_{r_2}} = \left((M^{(n)}_{\sigma_{r_1}})^{-1} - (M^{(n)}_{\sigma_{r_2}})^{-1}\right)^{-1}. \tag{22}$$

Hence we only need to construct $(M^{(n)}_{\sigma_r})^{-1}$, where $r \in \{2,\ldots,n-2,0\}$. For $q(\sigma)=\sigma_0$, it is easy to verify that the reduction matrix $M^{(n)}_{\sigma_0}$ is the $(n-3)!$-dimensional identity matrix. For $q(\sigma)=\sigma_r$, the Eq. (19) reads

$$(\gamma(r-2)+1)a_{\gamma(0),\ldots,\gamma(r-3),\gamma(r-2)+1,\gamma(r-1),\ldots,\gamma(n-4),0} = a_{\gamma(0),\ldots,\gamma(n-4),0}, \tag{23}$$

which holds for any $\gamma \in S_{n-3}$. The a-coefficient on the left-hand side above is *not* a canonical coefficient and can be rewritten in terms of the canonical ones as follows

$$a_{\gamma(0),\ldots,\gamma(r-3),\gamma(r-2)+1,\gamma(r-1),\ldots,\gamma(n-4),0} = \sum_{j=1}^{(n-3)!}\left(c_j^{\gamma(0),\ldots,\gamma(r-3),\gamma(r-2)+1,\gamma(r-1),\ldots,\gamma(n-4),0}\, a_{\gamma_j(0),\ldots,\gamma_j(n-4),1}\right), \tag{24}$$

where the coefficients c's are defined in (15) and are solely determined by (11). From this equation we read out the elements of $(M^{(n)}_{\sigma_r})^{-1}$ for $r\in[2,n-2]$ as follows

$$(M^{(n)}_{\sigma_r})^{-1}_{ij} = (\gamma_i(r-2)+1)c_j^{\gamma_i(0),\ldots,\gamma_i(r-3),\gamma_i(r-2)+1,\gamma_i(r-1),\ldots,\gamma_i(n-4),0}. \tag{25}$$

Using the reduction matrices repeatedly, any CHY-like integral with a completely factorized h_0 can be related to one with $h'_0=\sigma_r$.[5] The latter is the so-called prepared form and in [24] such CHY-like integrals with the one-loop scattering equations are studied. Here we repeat the exercise for the tree-level scattering equations. Let $a^{(\sigma_r)}$ denote the a-coefficients in the differential operator associated with the tree-level prepared form with $h_0=\sigma_r$. We obtain the following analytical expressions for the canonical ones

$$a^{(\sigma_r)}_{\gamma(0),\ldots,\gamma(n-4),0} = \begin{cases} \dfrac{\mathrm{sgn}(\gamma)(n-3)!}{\left(\partial^{\gamma(0)+1,\ldots,\gamma(r-3)+1,0,\gamma(r-1)+1,\ldots,\gamma(n-4)+1,1}(h_1h_2\ldots h_{n-3})\right)\Big|_{\sigma\to 0}}, & \text{for } \gamma(r-2)=0 \\ 0 & \text{for others} \end{cases},$$
$$a^{(\sigma_0)}_{\gamma(0),\ldots,\gamma(n-4),0} = \frac{\mathrm{sgn}(\gamma)(n-3)!}{\left(\partial^{\gamma(0)+1,\ldots,\gamma(n-4)+1,0}(h_1h_2\ldots h_{n-3})\right)\big|_{\sigma\to 0}}, \tag{26}$$

where $\mathrm{sgn}(\gamma)$ denotes the signature of the permutation γ. With the reduction matrices and the a-coefficients above, a generic CHY integral (7) can be evaluated straightforwardly

$$\mathcal{I}_n(P,h_0) = -\left[\mathbb{D}^{(m)}_{h_0}\frac{P(\sigma)}{(\sigma_0-1)}\right]\Big|_{\sigma\to 0} = \frac{-1}{(d_{h_0}-1)!} \times \sum_{i,j=1}^{(n-3)!}\left(M^{(n)}_{q^{(1)}}\ldots M^{(n)}_{q^{(d_{h_0}-1)}}\right)_{ij} a^{(\sigma_r)}_{\gamma_j(0),\ldots,\gamma_j(n-4),0} \left[\mathcal{D}^{(m)}_i\frac{P(\sigma)}{(\sigma_0-1)}\right]\Big|_{\sigma\to 0}. \tag{27}$$

3.3 Examples

Here we consider a couple of examples in detail to demonstrate the evaluation of tree-level CHY integrals, using the reduction matrix discussed above.

At four points, we have only one scattering equation after gauge fixing $\sigma_1\to 0, \sigma_3\to 1, \sigma_4\to\infty$. We consider the integral below as a simple example

$$\mathcal{I}_4(1,\sigma_2-1) = \oint_{h_1=0}\frac{d\sigma_2}{h_1}\frac{1}{\sigma_2-1} = \oint_{h_1=\sigma_0-1=0}\frac{d\sigma_2\wedge d\sigma_0}{h_1(\sigma_0-1)}\frac{1}{(\sigma_2-\sigma_0)\sigma_0}, \tag{28}$$

where in the second equal sign we have homogenized the scattering equation and the original denominator of the integrand. We have also used the trick of multiplying $\frac{1}{\sigma_0}$ to the integrand for the later use of the prepared form. The homogenized polynomial scattering equation reads

$$h_1 = s_{13}\sigma_2 + s_{12}\sigma_0. \tag{29}$$

We use the global residue theorem to change the integral contour around the solutions of $h_1=(\sigma_2-\sigma_0)\sigma_0=0$

$$\mathcal{I}_4(1,\sigma_2-1) = -\oint_{h_1=(\sigma_2-\sigma_0)\sigma_0=0}\frac{d\sigma_2\wedge d\sigma_0}{h_1(\sigma_2-\sigma_0)\sigma_0}\frac{1}{(\sigma_0-1)}, \tag{30}$$

This integral is then given by the action of a differential operator as follows

$$\mathcal{I}_4(1,\sigma_2-1) = -\left[\mathbb{D}^{(1)}_{(\sigma_2-\sigma_0)\sigma_0}\frac{1}{(\sigma_0-1)}\right]\Big|_{\sigma\to 0}, \tag{31}$$

[5] Even if h_0 does not have a factor σ_r, one can always multiply $\frac{\sigma_r}{\sigma_r}$ or $\frac{1}{\sigma_0}$ to the integrand and apply the reduction process to other factors.

where $\mathbb{D}^{(1)}_{(\sigma_2-\sigma_0)\sigma_0} = a_{1,0}\partial^{1,0} + a_{0,1}\partial^{0,1}$ and $\partial^{r_2,r_0} = (\frac{\partial}{\partial\sigma_2})^{r_2}(\frac{\partial}{\partial\sigma_0})^{r_0}$. The non-canonical $a_{1,0}$ is related to the canonical $a_{0,1}$ via (11) and we have

$$\mathbb{D}^{(1)}_{(\sigma_2-\sigma_0)\sigma_0} = a_{0,1}\mathcal{D}^{(1)}_1, \quad \text{with} \quad \mathcal{D}^{(1)}_1 = \partial^{0,1} - \frac{s_{12}}{s_{13}}\partial^{1,0}. \tag{32}$$

The reduction matrix $M^{(4)}_{\sigma_2-\sigma_0}$ can be obtained from (22) and (25) directly, which relates $a_{0,1}$ above to $a^{(\sigma_0)}_{0,0}$ given by (26). We have

$$\begin{aligned} M^{(4)}_{\sigma_2-\sigma_0} &= -\frac{s_{13}}{(s_{12}+s_{13})}, \\ a^{(\sigma_0)}_{0,0} &= \left[\frac{1}{\partial^{1,0}(h_1)}\right]\Bigg|_{\sigma\to 0} = \frac{1}{s_{13}}. \end{aligned} \tag{33}$$

Hence the integral reads simply

$$\begin{aligned} \mathcal{I}_4(1,\sigma_2-1) &= -M^{(4)}_{\sigma_2-\sigma_0}a^{(\sigma_0)}_{0,0}\left[\mathcal{D}^{(1)}_1\frac{\sigma_0}{(\sigma_0-1)}\right]\Bigg|_{\sigma\to 0} \\ &= -\frac{1}{s_{12}+s_{13}}. \end{aligned} \tag{34}$$

At five points, we consider below a typical CHY-like integral with a double pole and a nontrivial numerator

$$\begin{aligned} &\mathcal{I}_5((\sigma_2-1)\sigma_3,(\sigma_3-1)^2\sigma_2) \\ &= \oint_{\substack{h_1=h_2=0 \\ \sigma_0-1=0}} \frac{d\sigma_2\wedge d\sigma_3\wedge d\sigma_0}{h_1h_2(\sigma_0-1)}\frac{(\sigma_2-1)\sigma_3}{(\sigma_3-\sigma_0)^2\sigma_2}, \end{aligned} \tag{35}$$

where we have homogenized the scattering equations and the denominator $(\sigma_3-1)^2\sigma_2$ with the auxiliary variable σ_0. The numerator does not need to be homogenized for the application of our method. The homogenized polynomial scattering equations are

$$\begin{aligned} h_1 &= s_{134}\sigma_2 + s_{124}\sigma_3 + s_{123}\sigma_0 \\ h_2 &= s_{14}\sigma_2\sigma_3 + s_{13}\sigma_2\sigma_0 + s_{12}\sigma_3\sigma_0. \end{aligned} \tag{36}$$

After using the global residues theorem, the integral contour is changed to the circle around the solutions of $h_1 = h_2 = (\sigma_3-\sigma_0)^2\sigma_2 = 0$. This integral is then given by the action of the following differential operator

$$\begin{aligned} &\mathcal{I}_5((\sigma_2-1)\sigma_3,(\sigma_3-1)^2\sigma_2) \\ &= -\left[\mathbb{D}^{(3)}_{(\sigma_3-\sigma_0)^2\sigma_2}\frac{(\sigma_2-1)\sigma_3}{(\sigma_0-1)}\right]\Bigg|_{\sigma\to 0}. \end{aligned} \tag{37}$$

The differential operator takes the form below

$$\mathbb{D}^{(3)}_{(\sigma_3-\sigma_0)^2\sigma_2} = a_{0,0,3}\partial^{0,0,3} + a_{1,1,1}\partial^{1,1,1} + a_{0,1,2}\partial^{0,1,2} + \cdots, \tag{38}$$

where $\partial^{r_2,r_3,r_0} = (\frac{\partial}{\partial\sigma_2})^{r_2}(\frac{\partial}{\partial\sigma_3})^{r_3}(\frac{\partial}{\partial\sigma_0})^{r_0}$ and we have only written out the terms that have nonzero contributions. Here (11) gives the following equations

$$\begin{aligned} 6s_{45}a_{0,0,3} + 2s_{35}a_{0,1,2} + 2s_{25}a_{1,0,2} &= 0 \\ s_{235}a_{1,1,1} + 2s_{345}a_{0,1,2} + 2s_{245}a_{1,0,2} &= 0. \end{aligned} \tag{39}$$

The canonical coefficients here are $\{a_{0,1,2}, a_{1,0,2}\}$ and hence we have

$$\mathbb{D}^{(3)}_{(\sigma_3-\sigma_0)^2\sigma_2} = a_{0,1,2}\mathcal{D}^{(3)}_1 + a_{1,0,2}\mathcal{D}^{(3)}_2, \tag{40}$$

where

$$\begin{aligned} \mathcal{D}^{(3)}_1 &= -\frac{s_{35}}{3s_{45}}\partial^{0,0,3} - \frac{2s_{12}}{s_{14}}\partial^{1,1,1} + \partial^{0,1,2} + \cdots \\ \mathcal{D}^{(3)}_2 &= -\frac{s_{2,5}}{3s_{45}}\partial^{0,0,3} - \frac{2s_{13}}{s_{14}}\partial^{1,1,1} + \partial^{1,0,2} + \cdots. \end{aligned} \tag{41}$$

Applying the reduction matrix for the factor $\sigma_3-\sigma_0$ *twice*, the above canonical coefficients $\{a_{0,1,2}, a_{1,0,2}\}$ are related to the canonical coefficients $\{a^{(\sigma_2)}_{0,1,0}, a^{(\sigma_2)}_{1,0,0}\}$. Hence we only need to construct $M^{(5)}_{\sigma_3-\sigma_0}$. The reduction matrix for σ_0 is just two dimensional identity matrix. Recall the linearity property (22) and that $M^{(5)}_{\sigma_0}$ is simply a two-dimensional identity matrix. The only nontrivial ingredient here is $(M^{(5)}_{\sigma_3})^{-1}$. Recall that $M^{(5)}_{\sigma_3}$ is defined to relate an order-2 differential operator to an order-1 one. For $q = \sigma_3$, the relation (23) reads

$$a_{0,1,0} = 2a_{0,2,0}, \qquad a_{1,0,0} = a_{1,1,0}. \tag{42}$$

Moreover, the local duality conditions (11) for the a-coefficients in the order-2 differential operator read

$$\begin{aligned} 2s_{35}a_{0,2,0} + s_{25}a_{1,1,0} + s_{45}a_{0,1,1} &= 0, \\ s_{235}a_{1,1,0} + s_{345}a_{0,1,1} + s_{245}a_{1,0,1} &= 0. \end{aligned} \tag{43}$$

Plugging (42) in (43) we obtain

$$\begin{pmatrix} a_{0,1,0} \\ a_{1,0,0} \end{pmatrix} = \begin{pmatrix} -\frac{s_{35}}{s_{45}} & -\frac{s_{25}}{s_{45}} \\ \frac{s_{12}s_{35}}{s_{13}s_{45}} & \frac{s_{12}s_{25}-s_{14}s_{45}}{s_{13}s_{45}} \end{pmatrix} \begin{pmatrix} a_{0,1,1} \\ a_{1,0,1} \end{pmatrix}, \tag{44}$$

where the matrix is $(M^{(5)}_{\sigma_3})^{-1}$. With this matrix obtained, we can compute $M^{(5)}_{\sigma_3-\sigma_0}$ straightforwardly.

The canonical coefficients $\{a^{(\sigma_2)}_{0,1,0}, a^{(\sigma_2)}_{1,0,0}\}$ are again given by (26) as follows

$$a^{(\sigma_2)}_{0,1,0} = \frac{2}{\left(\partial^{0,2,1}(h_1h_2)\right)\Big|_{\sigma\to 0}} = \frac{1}{s_{12}s_{124}}, \qquad a^{(\sigma_2)}_{1,0,0} = 0. \tag{45}$$

Hence we have

$$\begin{aligned}\mathcal{I}_5 &= -\frac{1}{2}\sum_{i,j=1}^{2}\left(M^{(5)}_{\sigma_3-\sigma_0}M^{(5)}_{\sigma_3-\sigma_0}\right)_{i,j} a^{(\sigma_2)}_{\gamma_i(0),\gamma_i(1),0} \\ &\quad\times\left[\mathcal{D}^{(3)}_j\frac{1}{(\sigma_0-1)}\right]\Bigg|_{\sigma\to 0} \\ &= \frac{s_{234}s_{14}+s_{12}s_{24}+s_{234}s_{24}}{s_{12}s_{234}s_{34}^2},\end{aligned} \tag{46}$$

where $\gamma_i \in S_2$.

4 Gauge invariant BCJ numerators of Yang–Mills

In previous sections, we have presented general discussions on the evaluation of tree-level CHY-like integrals, using the differential operator and the reduction matrix. This approach only hinges on the scattering equations and the factorized form of the integrand and therefore applies to all theories whose amplitudes admit the CHY representations. In this section, we consider a particular application of the method: the construction of the BCJ numerators for tree-level Yang–Mills amplitudes.

4.1 BCJ numerators from CHY form

The CHY form [1–3] for a color-ordered amplitude in Yang–Mills at tree level reads

$$\begin{aligned}&A(1,\alpha,n-1,n) \\ &\quad= \oint_{h_1=\cdots=h_{n-3}=0} d\Omega_n\sigma^2_{1\,n-1\,n}\mathrm{PT}(1\,\alpha\,n-1\,n)\mathrm{Pf}'[\Psi_{1n}(\sigma)],\end{aligned} \tag{47}$$

where α denotes a permutation of $\{2,3,\ldots,n-2\}$. Here we have used the shorthand notation $\sigma_{i_1\ldots i_t} = (\sigma_{i_t}-\sigma_{i_1})\prod_{j=1}^{t-1}(\sigma_{i_j}-\sigma_{i_{j+1}})$ and introduced the measure

$$d\Omega_n = \frac{J_n(\sigma)\,d\sigma_2\wedge\cdots\wedge d\sigma_{n-2}}{h_1\cdots h_{n-3}}. \tag{48}$$

Recall that $J_n(\sigma) = \prod_{1\le r<t\le n-1}(\sigma_t-\sigma_r)$ denotes the Jacobian of the transformation to the polynomial scattering equations. We have adopted the gauge-fixing $\sigma_1\to 0, \sigma_{n-1}\to 1, \sigma_n\to\infty$ and $\sigma^2_{1\,n-1\,n}$ is the factor introduced after the gauge-fixing.

The Parke-Taylor factor corresponding to a given color ordering $(1,\alpha,n-1,n)$ reads

$$\mathrm{PT}(1\,\alpha\,n-1\,n) = \frac{1}{\sigma_{1\,\alpha(2)\ldots\alpha(n-2)\,n-1\,n}}. \tag{49}$$

$\mathrm{Pf}'[\Psi_{1n}(\sigma)]$ denotes the reduced Pfaffian of the matrix $\Psi(\sigma)$ with the first and the last columns and rows removed. The explicit expression for the reduced Pfaffian is given in [3] and there is a freedom of removing the i-th and j-th rows and columns of Ψ for any (i,j). Here for simplicity, we choose $(i,j)=(1,n)$.

On the other hand, the above color-ordered amplitudes is related to the BCJ numerators as follows,

$$A(1,\alpha,n-1,n) = \sum_{\beta\in S_{n-3}}\mathbf{m}(\alpha|\beta)N(1\,\beta\,n-1\,n), \tag{50}$$

where $\mathbf{m}(\alpha|\beta)$ denotes the propagator matrix [45], whose rows and columns are labeled by the color orderings $(1,\alpha,n-1,n)$ and $(1,\beta,n-1,n)$. The BCJ numerators $N(1\,\beta\,n-1\,n)$ are in the minimal basis. Similar to the color-ordered amplitude, the propagator matrix also admits a CHY-like representation

$$\begin{aligned}&\mathbf{m}(\alpha|\beta) \\ &\quad= -\oint_{h_1=\cdots=h_{n-3}=0}\sigma^2_{1\,n-1\,n}\mathrm{PT}(1\,\alpha\,n-1\,n)\mathrm{PT}(1\,\beta\,n-1\,n)d\Omega_n.\end{aligned} \tag{51}$$

Thus, comparing (47) and (51), we see that if $\mathrm{Pf}'[\Psi_{1n}(\sigma)]$ can be expanded in the basis spanned by the Parke-Taylor factors $\{\mathrm{PT}(1\,\beta\,n-1\,n)|\beta\in S_{n-2}\}$ with σ-independent coefficients, these coefficients are the BCJ numerators. To extract the BCJ numerators, we adopt a dual basis projector $\overline{\mathrm{PT}}(1\,\alpha\,n-1\,n)$ that satisfies the condition

$$\oint_{h_1=\cdots=h_{n-3}=0}(-1)\sigma^2_{1\,n-1\,n}\overline{\mathrm{PT}}(1\,\alpha\,n-1\,n)\mathrm{PT}(1\,\beta\,n-1\,n)d\Omega_n = \delta_{\alpha\beta}. \tag{52}$$

Recall that the propagator matrix $\mathbf{m}(\alpha|\beta)$ here is a $(n-3)!\times(n-3)!$ square matrix. In this case, $\mathbf{m}(\alpha|\beta)$ is invertible and its inverse is the KLT matrix $\mathcal{S}_{\alpha\beta}$ [37,46,47]. Thus the following is obviously the solution of the dual basis projector,

$$\overline{\mathrm{PT}}(1\,\alpha\,n-1\,n) = \sum_\beta\mathcal{S}_{\alpha\beta}\mathrm{PT}(1\,\beta\,n-1\,n). \tag{53}$$

Using the dual basis, the BCJ numerator is given as a CHY-like integral

$$N(1\,\alpha\,n-1\,n) = \oint_{h_1=\cdots=h_{n-3}=0} d\Omega_n\, \overline{\mathrm{PT}}(1\,\alpha\,n-1\,n)\sigma_{1n-1n}^2 \mathrm{Pf}'[\Psi_{1n}(\sigma)]\,, \tag{54}$$

The CHY-representations of BCJ numerators can be easily evaluated using the differential-operator based method. Generally speaking, such BCJ numerators in the minimal basis are all non-local. They may contain poles that do not correspond to the propagators in trivalent diagrams. But these unphysical poles do not contribute to the amplitudes. As we will observe in examples, the numerators computed this way are gauge invariant in $n-2$ legs and respect the crossing symmetry under the permutation of $n-3$ legs.

4.2 Yang–Mills BCJ numerator at four and five points

In this section, we construct the BCJ numerators at four and five points to further illustrate the application of the differential operator and the reduction matrix. As discussed above, the BCJ numerator can be computed by a CHY-like integral, which is readily evaluated by our differential-operator based method.

At four points, the minimal basis consists of only one independent BCJ numerator, which we choose to be $N(1234)$. In this case, (54) gives

$$N(1234) = \oint_{h_1=0} \sigma_{134}^2 \overline{\mathrm{PT}}(1234)\mathrm{Pf}'[\Psi_{14}(\sigma)] d\Omega_4\,, \tag{55}$$

where only one integration variable σ_2 is left after gauge fixing ($\sigma_1 \to 0, \sigma_3 \to 1, \sigma_4 \to \infty$). The KLT matrix is given in [37] and this leads to the dual basis projector below

$$\overline{\mathrm{PT}}(1234) = \frac{s_{12}s_{1,23}}{s_{13}}\mathrm{PT}(1234). \tag{56}$$

The reduced Pfaffian can be written as the following [48]

$$\begin{aligned}\mathrm{Pf}'[\Psi_{14}(\sigma)] = &-\frac{\epsilon_1\cdot F_2\cdot F_3\cdot\epsilon_4}{\sigma_{1234}} - \frac{\epsilon_1\cdot F_3\cdot F_2\cdot\epsilon_4}{\sigma_{1324}} \\ &+ \frac{\epsilon_1\cdot\epsilon_4 \mathrm{tr}(F_2\cdot F_3)}{2\sigma_{14}^2\sigma_{23}^2} + \frac{\epsilon_1\cdot F_2\cdot\epsilon_4}{\sigma_{124}}C_{(3)} \\ &+ \frac{\epsilon_1\cdot F_3\cdot\epsilon_4}{\sigma_{134}}C_{(2)} - \frac{\epsilon_1\cdot\epsilon_4}{\sigma_{14}^2}C_{(2)}C_{(3)}\,,\end{aligned} \tag{57}$$

where we have adopted the following notations also used in [48]

$$F_i = k_i^\mu \epsilon_i^\nu - \epsilon_i^\mu k_i^\nu\,, \qquad C_{(i)} = \sum_{j=1, j\neq i}^{4} \frac{k_j\cdot\epsilon_i}{\sigma_j-\sigma_i}\,. \tag{58}$$

We now demonstrate the evaluation of (55) using the reduction matrix. We consider the term below as an example and all other terms can be computed in the same way,

$$N_4 \equiv \oint_{h_1=\sigma_0-1=0} \sigma_{134}^2\, \overline{\mathrm{PT}}(1234)\, \frac{\epsilon_1\cdot F_2\cdot\epsilon_4}{\sigma_{124}}\, C_{(3)}\, d\Omega_4\,. \tag{59}$$

Plugging in (56) and homogenizing the scattering equations and the denominators, we have

$$\begin{aligned}N_4 = &\frac{s_{12}s_{1,23}\epsilon_1\cdot F_2\cdot\epsilon_4}{s_{13}} \oint_{\substack{h_1=0\\ \sigma_0-1=0}} \frac{d\sigma_2\wedge d\sigma_0}{(\sigma_2-\sigma_0)(\sigma_0-1)} \\ &\times\left(\frac{\epsilon_3\cdot k_1}{\sigma_0} - \frac{\epsilon_3\cdot k_1}{\sigma_2} - \frac{\epsilon_3\cdot k_2}{\sigma_2}\right).\end{aligned} \tag{60}$$

Each term in the expression above equals to the action of a first-order differential operator, which can be written in terms of a common reduction matrix $M^{(4)}_{\sigma_2-\sigma_0}$ and the canonical coefficients of two prepared forms. That is,

$$\begin{aligned}N_4 = &\frac{-s_{12}s_{1,23}\epsilon_1\cdot F_2\cdot\epsilon_4}{s_{13}} \\ &\times M^{(4)}_{\sigma_2-\sigma_0}\left(\epsilon_3\cdot k_1 a_{0,0}^{(\sigma_0)} - \epsilon_3\cdot k_1 a_{0,0}^{(\sigma_2)} - \epsilon_3\cdot k_2 a_{0,0}^{(\sigma_2)}\right) \\ &\times\left[\mathcal{D}_1^{(1)}\frac{1}{\sigma_0-1}\right]\Bigg|_{\sigma\to 0}\,,\end{aligned} \tag{61}$$

where $\mathcal{D}_1^{(1)} = \partial^{0,1} - \frac{s_{12}}{s_{13}}\partial^{1,0}$, $M^{(4)}_{\sigma_2-\sigma_0}$ has been computed previously in (33) and $a_{0,0}^{(\sigma_r)}$ is given by (26). Explicitly, they read

$$M^{(4)}_{\sigma_2-\sigma_0} = -\frac{s_{13}}{s_{1,23}}, \quad a_{0,0}^{(\sigma_2)} = -\frac{1}{s_{12}}, \quad a_{0,0}^{(\sigma_0)} = \frac{1}{s_{13}}. \tag{62}$$

Evaluating other terms in (57) similarly, we obtain the 4-point BCJ numerator in the minimal basis

$$\begin{aligned}N(1234) = &\epsilon_1\cdot F_2\cdot F_3\cdot\epsilon_4 + \frac{s_{12}}{s_{13}}\epsilon_1\cdot F_3\cdot F_2\cdot\epsilon_4 \\ &- \frac{s_{12}}{s_{1,23}}\mathrm{tr}(F_2\cdot F_3)\epsilon_1\cdot\epsilon_4 - \frac{1}{s_{13}}k_1\cdot F_3\cdot k_2\epsilon_1\cdot F_2\cdot\epsilon_4 \\ &- \frac{1}{s_{13}}k_1\cdot F_2\cdot k_3\,\epsilon_1\cdot F_3\cdot\epsilon_4 \\ &- \frac{1}{s_{13}s_{1,23}}k_1\cdot F_3\cdot k_2 k_1\cdot F_2\cdot k_3\epsilon_1\cdot\epsilon_4\,.\end{aligned} \tag{63}$$

As F_i vanishes under the gauge transformation $\epsilon_i \to k_i$, $N(1234)$ is manifestly gauge invariant in leg 2 and leg 3.

At five points, the minimal basis consists of two independent BCJ numerators $N(12345)$ and $N(13245)$. As pointed out before, the numerators computed this way is crossing symmetric and therefore it suffices to compute $N(12345)$ only. This numerator reads

$$N(12345)=\oint_{h_1=h_2=0}\sigma_{145}^2\,\overline{\mathrm{PT}}(12345)\mathrm{Pf}'[\Psi_{15}(\sigma)]d\Omega_5\,. \tag{64}$$

where the dual basis projector $\overline{\mathrm{PT}}(12345)$ can be easily computed from (53) and takes the form below

$$\begin{aligned}\overline{\mathrm{PT}}(12345)&=\left(\frac{s_{12}s_{34}s_{1,24}s_{12,3}}{s_{14}s_{124}}+\frac{s_{12}^2s_{34}s_{13}}{s_{14}s_{134}}\right)\mathrm{PT}(12345)\\&+\left(\frac{s_{12}s_{13}s_{24}s_{34}}{s_{14}s_{134}}+\frac{s_{12}s_{13}s_{24}s_{34}}{s_{14}s_{124}}\right)\mathrm{PT}(13245).\end{aligned} \tag{65}$$

$$M^{(5)}_{C_{(4)}}\equiv\begin{pmatrix}\frac{-s_{1,34}s_{124}k_2\cdot F_4\cdot k_3+s_{234}s_{34}k_1\cdot F_4\cdot k_2}{s_{234}s_{24}s_{34}} & -\frac{s_{13}s_{134}k_2\cdot F_4\cdot k_3}{s_{234}s_{24}s_{34}}\\ -\frac{s_{12}s_{124}k_3\cdot F_4\cdot k_2}{s_{234}s_{24}s_{34}} & \frac{-s_{1,24}s_{134}k_3\cdot F_4\cdot k_2+s_{234}s_{24}k_1\cdot F_4\cdot k_3}{s_{234}s_{24}s_{34}}\end{pmatrix} \tag{69}$$

The reduced Pfaffian at five points reads [48]

$$\begin{aligned}&\mathrm{Pf}'[\Psi_{15}(\sigma)]\\&=\sum_{\gamma\in S_3}\left(-\frac{\epsilon_1\cdot F_{\gamma(2)}\cdot F_{\gamma(3)}\cdot F_{\gamma(4)}\cdot\epsilon_5}{\sigma_{1\gamma(2)\gamma(3)\gamma(4)5}}\right.\\&\left.+\frac{\epsilon_1\cdot F_{\gamma(2)}\cdot\epsilon_5\mathrm{tr}(F_{\gamma(3)}F_{\gamma(4)})}{4\sigma_{1\gamma(2)5}\sigma_{\gamma(3)\gamma(4)}}\right)+\frac{\epsilon_1\cdot\epsilon_5\mathrm{tr}(F_2F_3F_4)}{2\sigma_{15}\sigma_{234}}\\&+\frac{\epsilon_1\cdot\epsilon_5\mathrm{tr}(F_3F_2F_4)}{2\sigma_{15}\sigma_{324}}+\sum_{\gamma\in S_3}\left(\frac{\epsilon_1\cdot F_{\gamma(2)}\cdot F_{\gamma(3)}\cdot\epsilon_5C_{(\gamma(4))}}{\sigma_{1\gamma(2)\gamma(3)5}}\right.\\&\left.-\frac{\epsilon_1\cdot F_{\gamma(2)}\cdot\epsilon_5C_{(\gamma(3))}C_{(\gamma(4))}}{2\sigma_{1\gamma(2)5}}\right)\\&-\epsilon_1\cdot\epsilon_5\left(\frac{\mathrm{tr}(F_2F_3)C_{(4)}}{\sigma_{15}\sigma_{23}}+\frac{C_{(2)}\mathrm{tr}(F_3F_4)}{\sigma_{15}\sigma_{34}}\right.\\&\left.+\frac{\mathrm{tr}(F_2F_4)C_{(3)}}{\sigma_{15}\sigma_{24}}-\frac{C_{(2)}C_{(3)}C_{(4)}}{\sigma_{15}}\right).\end{aligned} \tag{66}$$

Like the four-point case, we also consider the characteristic term below as an example here and the rest can all be computed in the same way,

$$N_5=\oint_{h_1=h_2=0}\sigma_{145}^2\,\overline{\mathrm{PT}}(12345)\frac{\epsilon_1\cdot F_2\cdot F_3\cdot\epsilon_5C_{(4)}}{\sigma_{1235}}d\Omega_5. \tag{67}$$

Taking $\sigma_1\to0$, $\sigma_4\to1$, $\sigma_5\to\infty$, we are only left with two variables σ_2 and σ_3. Thescattering equations and the denominators above are homogenized with an auxiliary variable σ_0. This process leads to

$$\begin{aligned}C_{(4)}&=\frac{s_{14}}{-\sigma_0}+\frac{s_{24}}{\sigma_2-\sigma_0}+\frac{s_{34}}{\sigma_3-\sigma_0}\,,\\\frac{1}{\sigma_{1235}}&=\frac{1}{\sigma_2(\sigma_2-\sigma_3)}\,.\end{aligned} \tag{68}$$

There are no other denominators in (67) after gauge fixing. The factor $C_{(4)}$ can now be replaced by a matrix denoted as $M^{(5)}_{C_{(4)}}$, which is a sum over three reduction matrices, each corresponding to the denominator of one of its three terms. These reduction matrices can all be constructed following the discussions in the previous section. This matrix $M^{(5)}_{C_{(4)}}$ reads

Another common reduction matrix $M^{(5)}_{\sigma_2-\sigma_3}$ renders (67) a prepared form. This reduction matrix reads

$$M^{(5)}_{\sigma_2-\sigma_3}=\begin{pmatrix}\frac{(s_{12}s_{14}+s_{1,34}s_{23})s_{124}}{s_{1,234}s_{23}s_{123}} & \frac{s_{13}(s_{14}-s_{23})s_{134,2}}{s_{1,234}s_{23}s_{123}}\\ \frac{s_{12}(s_{14}-s_{23})s_{124}}{s_{1,234}s_{23}s_{123}} & \frac{(s_{13}s_{14}+s_{1,24}s_{23})s_{134,2}}{s_{1,234}s_{23}s_{123}}\end{pmatrix}. \tag{70}$$

The canonical a-coefficients for the prepared form with $h_0=\sigma_2$ are

$$a_{010}^{(\sigma_2)}=\frac{1}{s_{12}s_{124}},\qquad a_{100}^{(\sigma_2)}=0. \tag{71}$$

Hence (67) is given by the action of the differential operator as follows

$$\begin{aligned}N_5&=-\frac{\epsilon_1\cdot F_2\cdot F_3\cdot\epsilon_5}{2}\sum_{i,j=1}^{2}\left(M^{(5)}_{C_{(4)}}M^{(5)}_{\sigma_2-\sigma_3}\right)_{i,j}a^{(\sigma_2)}_{\gamma_i(0),\gamma_i(1),0}\\&\times\left[\mathcal{D}_j^{(3)}\frac{\sigma_{145}\,\overline{\mathrm{PT}}(12345)J_5(\sigma)}{(\sigma_0-1)}\right]\Bigg|_{\sigma\to0}\,,\end{aligned} \tag{72}$$

where $\mathcal{D}_j^{(3)}$ is given in (41) and $\gamma_i\in S_2$. Plugging in the expressions (41), (69), (70) and (71), we obtain

$$N_5=\frac{k_1\cdot F_4\cdot k_3}{s_{14}}\epsilon_1\cdot F_2\cdot F_3\cdot\epsilon_5-\frac{s_{34}k_1\cdot F_4\cdot k_2}{s_{14}s_{124}}\epsilon_1\cdot F_2\cdot F_3\cdot\epsilon_5. \tag{73}$$

All other terms in (66) are treated similarly and the explicit expression for the five-point BCJ numerator reads

$$\begin{aligned}
N(12345) =& \left(k_1\cdot F_4\cdot k_3 - \frac{s_{34}}{s_{124}} k_1\cdot F_4\cdot k_2\right) \frac{\epsilon_1\cdot F_2\cdot F_3\cdot\epsilon_5}{s_{14}} \\
&+ \frac{1}{s_{14}} k_1\cdot F_2\cdot F_3\cdot F_4\cdot k_1 \epsilon_1\cdot\epsilon_5 \\
&+ \left(\frac{1}{s_{134}} k_1\cdot F_2\cdot k_{34} k_1\cdot F_3\cdot F_4\cdot k_1\right. \\
&\left.+ \frac{1}{s_{124}} k_{12}\cdot F_3\cdot k_4 k_1\cdot F_2\cdot F_4\cdot k_1\right) \frac{\epsilon_1\cdot\epsilon_5}{s_{14}} \\
&+ \left(\frac{s_{12}s_{23} + s_{14}s_{2,34}}{s_{124}s_{14}s_{134}} k_1\cdot F_3\cdot F_4\cdot k_1\right. \\
&\left.+ \frac{s_{12}s_{23} + s_{12,3}s_{2,34}}{s_{25}s_{34}} \mathrm{tr}(F_3\cdot F_4)\right) \epsilon_1\cdot F_2\cdot\epsilon_5 \\
&- \left(k_1\cdot F_3\cdot F_4\cdot k_2 + \frac{s_{1,24}}{s_{14}} k_2\cdot F_3\cdot F_4\cdot k_1\right. \\
&\left.+ k_2\cdot F_3\cdot F_4\cdot k_2\right) \frac{\epsilon_1\cdot F_2\cdot\epsilon_5}{s_{124}} \\
&+ \left(\frac{(s_{14} - s_{23})\, s_{34}}{s_{124}s_{14}} k_1\cdot F_2\cdot F_4\cdot k_1 - k_1\cdot F_2\cdot F_4\cdot k_3\right. \\
&\left.- \frac{1}{s_{14}} k_1\cdot F_2\cdot k_3\, k_1\cdot F_4\cdot k_3\right) \frac{\epsilon_1\cdot F_3\cdot\epsilon_5}{s_{134}} \\
&+ \left(k_1\cdot F_2\cdot F_3\cdot k_4 + \frac{s_{12}}{s_{14}} k_{14}\cdot F_2\cdot F_3\cdot k_4\right. \\
&\left.- \frac{s_{1,24}}{s_{14}} k_1\cdot F_2\cdot k_3\, k_1\cdot F_3\cdot k_4\right) \frac{\epsilon_1\cdot F_4\cdot\epsilon_5}{s_{124}} \\
&+ \left(\frac{s_{12}}{s_{14}s_{134}} k_3\cdot F_2\cdot k_4\, k_1\cdot F_3\cdot k_4\right. \\
&\left.- \frac{1}{s_{134}} k_1\cdot F_2\cdot k_4\, k_1\cdot F_3\cdot k_4\right) \frac{\epsilon_1\cdot F_4\cdot\epsilon_5}{s_{124}} \\
&+ \frac{s_{12}}{s_{14}s_{134}} k_1\cdot F_4\cdot k_3\, \epsilon_1\cdot F_3\cdot F_2\cdot\epsilon_5 \\
&+ \frac{1}{s_{124}} k_{12}\cdot F_3\cdot k_4\, \epsilon_1\cdot F_2\cdot F_4\cdot\epsilon_5 \\
&+ \left(\frac{(s_{14} - s_{23})}{s_{134}} k_1\cdot F_3\cdot k_4 + k_2\cdot F_3\cdot k_4\right) \frac{s_{12}\epsilon_1\cdot F_4\cdot F_2\cdot\epsilon_5}{s_{14}s_{124}} \\
&- \left(\frac{s_{1,24}s_{13,4}}{s_{14}} k_1\cdot F_2\cdot k_3 - s_{13} k_1\cdot F_2\cdot k_4\right. \\
&\left.- \frac{s_{12}s_{13,4}}{s_{14}} k_3\cdot F_2\cdot k_4\right) \frac{\epsilon_1\cdot F_4\cdot F_3\cdot\epsilon_5}{s_{124}s_{134}} \\
&+ \frac{1}{s_{134}} k_1\cdot F_2\cdot k_{34}\, \epsilon_1\cdot F_3\cdot F_4\cdot\epsilon_5 - \epsilon_1\cdot F_2\cdot F_3\cdot F_4\cdot\epsilon_5 \\
&- \frac{s_{12,3}}{s_{124}} \epsilon_1\cdot F_2\cdot F_4\cdot F_3\cdot\epsilon_5 \\
&- \frac{s_{12}}{s_{134}} \epsilon_1\cdot F_3\cdot F_4\cdot F_2\cdot\epsilon_5 + \frac{s_{12}s_{34}}{s_{124}s_{14}} \epsilon_1\cdot F_4\cdot F_2\cdot F_3\cdot\epsilon_5 \\
&+ \frac{s_{12}s_{13,4}}{s_{134}s_{14}} \epsilon_1\cdot F_4\cdot F_3\cdot F_2\cdot\epsilon_5. \qquad (74)
\end{aligned}$$

The other numerator is obtained by the permuting indices

$$N(13245) = N(12345)|_{2\leftrightarrow 3}.$$

These non-local BCJ numerators are compact and again manifestly gauge invariant in the leg 2,3 and 4, due to the existence of the F_i factors.

5 Conclusion and outlooks

In this paper, we have developed a systematic method to construct the BCJ numerators, starting from the CHY forms of scattering amplitudes and using the differential operator proposed in our previous work [23]. This method is based on the key observation that the number of canonical coefficients in such a differential operator is always $(n-3)!$ for a n-point CHY form, independent of the order of the operator. In the process of solving for the canonical coefficients, we have built the reduction matrices to simplify the differential operator and improve the efficiency of computation. The reduction matrices are also universal for all the theories. In the end, as we have demonstrated, we always arrive at the prepared forms for which the coefficients are solved analytically in [24].

Both the BCJ numerators and the amplitudes obtained this way enjoy the manifest gauge invariance in $(n-2)$ out of the n external legs, and their final expressions are always of factorized forms. It is hopeful to formulate a closed form for the reduction matrix in future works, since a given reduction matrix M_{σ_i} is determined only by the factor σ_i and the polynomial scattering equations. Moreover, the polynomial scattering equations have nice combinatoric structures to exploit, which may allow us to produce analytical results for a general reduction matrix. As discussed in [21], the polynomial scattering equations is a Macaulay H-basis. This property may be helpful to prove the observation of the number $(n-3)!$ of independent a-coefficients.

In this paper, the concept of dual basis is adopted to extract the non-local BCJ numerators in the minimal basis, in which the non-local propagators can be removed using the BCJ relations. It is conceivable that similar projectors can be constructed for the local BCJ numerators as well. It can be expected that such projectors are constructed recursively, which may point to novel algebraic structures [49–52].

Our method can be easily generalized to loop levels. The one-loop scattering equations and the prepared forms have been studied in [24]. We expect the canonical coefficients can also be found at one loop and then the construction of reduction matrix is expected to be straightforward. Another future direction is to carry our method over to string theory. String amplitudes, written in the forms of worldsheet integrals, have a number of features in common with the CHY-like integrals. It is reasonable to hope that there exist similar differential operators and even reduction matrices which can help us evaluate those worldsheet integrals efficiently while preserving the factorized form.

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